ON SHAPE OPTIMIZATION AND THE POMPEIU PROBLEM

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ABSTRACT. The Pompeiu problem is considered as shape optimization problem. We show stability of the ball which is the minimum point of related domain functional. The proof is based on shape derivative method. Stability of the ball for general domain functionals invariant under the rigid motions is discussed.

INTRODUCTION AND STATEMENTS

Energy–type functionals $F[\omega] = \iint f(|x-y|)\, d\omega(x)\, d\omega(y)$, where ω is a measure of compact support, appears in statistical mechanics of systems, topological classification problems of knots, isoperimetric problems, harmonic analysis, discrete energy problems and other areas of pure and applied mathematics (see, for example, [6, 9, 17, 4, 15]).

Motivated by problems of shape optimal design [7, 23, 12, 14, 10] we consider the model domain functional

$$(0.1) \qquad F[\Omega] = \int_\Omega \int_\Omega f(|x-y|)\, dx\, dy,$$

defined on the set of bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with smooth boundaries. Here $f : (0, \infty) \to \mathbb{R}$ is known function.

Let $V \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^n)$. Equation

$$(0.2) \qquad \dot{x} = V(x)$$

will produce a flow $T_t^V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which moves a bounded smooth domain $\Omega = \Omega_0$ to its new position $\Omega_t^V = T_t^V(\Omega_0)$ (see Fig. 1 on p. 5). For all t close enough to zero the domain Ω_t^V has been shown to be bounded and smooth [23].

Shape derivative in the direction of vector field V is defined by formula

$$(0.3) \qquad \dot{F}[\Omega_0; V] := \frac{d}{dt}\big|_{t=0} F[\Omega_t^V].$$

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Definition 0.1. We say nonempty domain Ω is *critical* if

$$\dot{F}\left[\Omega; V\right] = 0$$

for any $V \in \mathrm{C}^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$.

If Ω is critical, then (see Section 1) equality

$$(0.4) \qquad \int_{\Omega} f\left(|x - y|\right) \, \mathrm{d}y = const$$

holds for all $x \in \partial\Omega$ with $const = 0$.

As a special case equation (0.4) includes the inverse potential problem

$$(0.5) \qquad \int_{\Omega} |x - y|^{2-n} \, \mathrm{d}y = const.$$

This last equation defines the body which has constant gravity potential on its own shape. The observation based on the "moving planes" method was done in [11]:

Proposition 0.2. *Let* C^1 *function f is strictly monotone. If nonempty* C^1 *domain Ω solves equation (0.4), then it is a ball.*

Thus, if C^1 function f is strictly monotone, then any critical point of the domain functional (0.1) is the ball of fixed radius. If function f is not strictly monotone, then critical domains of different shapes are possible.

Usually *stable* minimum point of general domain functional F is defined as the domain Ω_0 at which inequality $F\left[\Omega_0\right] < F\left[\Omega\right]$ holds for all Ω close enough to Ω_0 (see, for example, [8]). Even despite of the difficulty in proper understanding of the closeness, domain functional (0.1) has no stable minimum (and maximum) points in this sense because it is invariant under the rigid motions.

To reach the uniformity with definition of critical point we shall introduce the notion of *weakly stable* minimum point for the domain functional F.

Definition 0.3. We say the domain $\Omega = \Omega_0$ is stable minimum point of the domain functional (0.1) in the direction of vector field $V \in \mathrm{C}^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$, if

either for all t, $t \neq 0$ and close enough to zero, inequality $F\left[\Omega_0\right] < F\left[\Omega_t^V\right]$ holds,

either for all t close enough to zero, Ω_0 as the rigid body coincides with Ω_t^V.

The domain $\Omega = \Omega_0$ is said to be *weakly stable* minimum point of the domain functional (0.1) in $\mathrm{C}^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$, if this domain is stable minimum point of F in the direction of vector field $V \in \mathrm{C}^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$ for any vector field $V \in \mathrm{C}^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$.

Remark 0.4. a) Inequality

$$\ddot{F}\left[\Omega_0; V, V\right] > 0,\tag{0.6}$$

here $\ddot{F}$ denotes the second shape derivative of the domain functional F, is sufficient for the critical point $\Omega = \Omega_0$ to be stable minimum in the direction of vector field V.

b) If $V = const$, then any Ω_0 as the rigid body coincides with Ω_t^V for all $t \in \mathbb{R}$.

c) If normal part of V on the boundary $\partial\Omega_0$ is zero, then $\Omega_0 = \Omega_t^V$ for all t close to zero.

Statements **b)** and **c)** are easy to understand from geometrical point of view. They follow from the standard properties of ordinary differential equations.

If C^1 function f is strictly monotone, then by Proposition 0.2 the domain functional F has at most one critical point to be considered as a rigid body which, of course, is weakly stable in $\mathrm{C}^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$.

In this paper we consider domain functional (0.1) when function $f\left(|\cdot|\right)$ is positive definite, that is

$$\iint f\left(|x - y|\right) \mu\left(x\right) \bar{\mu}\left(y\right) \, \mathrm{d}x \, \mathrm{d}y \geq 0\tag{0.7}$$

for all complex $\mu \in \mathrm{C}_0^\infty\left(\mathbb{R}^n\right)$ and thus inequality

$$F\left[\Omega\right] \geq 0\tag{0.8}$$

holds for all Ω bounded.

Theorem 0.5. *Let F is the domain functional (0.1) generated by positive definite function $f\left(|\cdot|\right)$ and*

$$F\left[\Omega\right] = 0\tag{0.9}$$

for some Ω the ball.

Then Ω is weakly stable minimum point of F in $\mathrm{C}^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$.

Investigation of the domain functional (0.1) with the positive definite function $f\left(|\cdot|\right)$ is stimulated because of its relation to the Pompeiu problem [22]. We will prove

Proposition 0.6. *Let Ω be bounded nonempty domain.*
a) *If Ω satisfies (0.9) for some positive definite function $f\left(|\cdot|\right)$, then there is $\lambda > 0$:*

$$\hat{\chi}_\Omega\left(\xi\right) = 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n, \quad |\xi| = \lambda,\tag{0.10}$$

here $\hat{\chi}_\Omega$ denotes the Fourier transform of indicator function χ_Ω.

b) *On the other hand, if condition (0.10) holds, then $F\left[\Omega\right] = 0$ for some positive definite function $f\left(|\cdot|\right)$.*

Domain $\Omega \subset \mathbb{R}^n$ which satisfies (0.10) is known as domain without the *Pompeiu property* [5]. If its boundary $\partial\Omega$ is smooth and connected, then overdetermined problem

$$(0.11a) \qquad\qquad \Delta u + \lambda^2 u = 0 \quad \text{in} \quad \Omega$$

$$(0.11b) \qquad\qquad\qquad\qquad u = 1 \quad \text{on} \quad \partial\Omega$$

$$(0.11c) \qquad\qquad\qquad\qquad \nabla u = 0 \quad \text{on} \quad \partial\Omega$$

has nontrivial solution [2, 20]. It is expected (Schiffer conjecture [21]) such the domain to be necessarily a ball.

In [1, 13] the Schiffer conjecture is affirmed for "small" perturbations of the ball. Kobayashi [13] use precise Fourier transform estimates of domain indicator function. Agranovsky and Semenov [1], instead, exploit the overdetermined problem (0.11). Their approach leads to the generalization of results in Riemannian spaces. More on the Pompeiu and Schiffer problems see the expository paper [3].

The proof of the Theorem 0.5 may be derived from [1, 13]. Instead of, we consider the Pompeiu problem as shape optimization problem and shall give the direct proof of the Theorem 0.5 based on the shape derivative method.

The paper is organized as follows. In Section 1 shape derivatives of the domain functional (0.1) are calculated. The domain equation (0.4) which describes critical domains is derived and an expression of the second shape derivative of the domain functional (0.1) in a case of positive definite function $f(|\cdot|)$ is examined.

The proof of the Theorem 0.5 is preceded by

Lemma 0.7. *Let F is the domain functional (0.1) generated by positive definite function $f(|\cdot|)$ and $F[\Omega] = 0$ for Ω the ball centered at $x = 0$. Then $\ddot{F}[\Omega; V, V] = 0$ for vector field V if and only if the corresponding function $v = \langle V, x \rangle$ on the boundary of Ω coincides with first order spherical harmonic.*

This lemma is proved in Section 2.

Based on the Lemma 0.7 the proof of Theorem 0.5 is done in Section 3 and consists of two steps. First, we consider special choice of vector fields V when all domains Ω_t^V has the same center of mass $x = 0$ (Lemma 3.1). Second, the case when center of mass moves, is reduced to the previous step by suitable choice of non–autonomous vector field.

The proof of the Proposition 0.6 about the Pompeiu problem equivalence to the shape optimization problem is done in Section 4.

In Section 5 the remark on generalization of Theorem 0.5 to other domain functionals is done. Then a few numerical examples of shape evolution to the solution of the Pompeiu problem are presented.
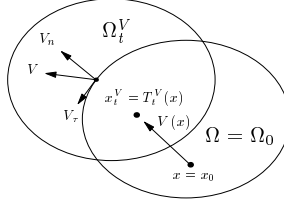
FIGURE 1.

1. SHAPE DERIVATIVES OF THE DOMAIN FUNCTIONAL

Let $f : (0, \infty) \to \mathbb{R}$ be the smooth function. We consider the domain functional

$$
F[\Omega] = \int_\Omega \int_\Omega f(|x - y|) \, \mathrm{d}x \, \mathrm{d}y. \tag{1.1}
$$

Then, according to notions on page 1 (see also Fig. 1),

$$
F[\Omega_t^V] = \int_{\Omega_0} \int_{\Omega_0} f(|T_t^V(x) - T_t^V(y)|) \det DT_t^V(x) \det DT_t^V(y) \, \mathrm{d}x \, \mathrm{d}y, \tag{1.2}
$$

here $\det DT_t^V$ denotes the Jacobian of the mapping $T_t^V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, t considered as parameter.

Using equalities

$$
\frac{d}{dt} T_t^V(x) = V(T_t^V(x)), \tag{1.3}
$$

$$
T_0^V(x) = x, \tag{1.4}
$$

$$
\frac{d}{dt}\Big|_{t=0} \det DT_t^V(x) = \mathrm{div}\, V(x), \tag{1.5}
$$

(see [23] on calculus technique) and verifying the equality

$$
\frac{d}{dt}\Big|_{t=0} f(|T_t^V(x) - T_t^V(y)|) = \tag{1.6}
$$
$$
\langle \nabla_x f(|x-y|), V(x) \rangle + \langle \nabla_y f(|x-y|), V(y) \rangle
$$

one gets

$$
\frac{d}{dt}\Big|_{t=0} F[\Omega_t^V] = \int_{\Omega_0} \int_{\Omega_0} \{[\langle \nabla_x f(|x-y|), V(x) \rangle + \langle \nabla_y f(|x-y|), V(y) \rangle] \tag{1.7}
$$
$$
+ f(|x-y|)[\mathrm{div}\, V(x) + \mathrm{div}\, V(y)]\} \, \mathrm{d}x \, \mathrm{d}y =
$$
$$
2 \int_{\Omega_0} \int_{\Omega_0} \mathrm{div}_x [f(|x-y|) V(x)] \, \mathrm{d}x \, \mathrm{d}y = 2 \int_{\partial\Omega_0} \left[\int_{\Omega_0} f(|x-y|) \, \mathrm{d}y \right] \langle V, \eta \rangle(x) \, \mathrm{d}s_x,
$$

here η is exterior unit normal vector field on $\partial\Omega_0$.

If Ω_0 gives an extremum to F, then necessarily

$$(1.8) \qquad \int_{\Omega_0} f\left(|x-y|\right) \, \mathrm{d}y = 0$$

for all $x \in \partial\Omega_0$.

Replacement of the "initial moment" $t = 0$ by the arbitrary moment t leads to equality

$$(1.9) \qquad \frac{d}{dt} F\left[\Omega_t^V\right] = 2 \int_{\partial\Omega_t^V} \left[\int_{\Omega_t^V} f\left(|x-y|\right) \, \mathrm{d}y \right] \left\langle V, \eta_t^V \right\rangle (x) \, \mathrm{d}s_x,$$

here η_t^V denotes exterior unit normal vector field on $\partial\Omega_t^V$.

Now let us calculate the second derivative

$$(1.10) \qquad \ddot{F}\left[\Omega_0; V, V\right] := \frac{d^2}{dt^2}|_{t=0} F\left[\Omega_t^V\right]$$

in case when Ω_0 is critical domain to functional F.

Applying formula (5.9) in page 1105 of [23] and because of (1.8) one gets

(1.11)

$$\frac{1}{2}\ddot{F}\left[\Omega_0; V, V\right] = \frac{d}{dt}|_{t=0} \left\{ \int_{\partial\Omega_t^V} \left[\int_{\Omega_t^V} f\left(|x-y|\right) \, \mathrm{d}y \right] \left\langle V, \eta_t^V \right\rangle (x) \, \mathrm{d}s_x \right\} =$$

$$\int_{\partial\Omega_0} \left\{ \frac{d}{dt}|_{t=0} \left[\int_{\Omega_0} f\left(|T_t^V(x) - T_t^V(y)|\right) \det DT_t^V(y) \, \mathrm{d}y \right] \left\langle V, \eta_t^V \right\rangle \left(T_t^V(x)\right) \right.$$

$$\left. + (n-1) K(x) \int_{\Omega_0} f\left(|x-y|\right) \, \mathrm{d}y \cdot \left\langle V, \eta \right\rangle^2 (x) \right\} \mathrm{d}s_x =$$

$$\int_{\partial\Omega_0} \left\{ \int_{\Omega_0} \left[\mathrm{div}_y \left(f\left(|x-y|\right) V(y) \right) + \left\langle \nabla_x f\left(|x-y|\right), V(x) \right\rangle \right] \mathrm{d}y \right\} \left\langle V, \eta \right\rangle (x) \, \mathrm{d}s_x =$$

$$\int_{\partial\Omega_0} \int_{\partial\Omega_0} f\left(|x-y|\right) \left\langle V, \eta \right\rangle (x) \left\langle V, \eta \right\rangle (y) \, \mathrm{d}s_x \, \mathrm{d}s_y$$

$$+ \int_{\partial\Omega_0} \left\langle \nabla_x \int_{\Omega_0} f\left(|x-y|\right) \, \mathrm{d}y, V(x) \right\rangle \left\langle V, \eta \right\rangle (x) \, \mathrm{d}s_x,$$

here $K(x)$ denotes mean curvature of $\partial\Omega_0$ at the point x.

If function $f\left(|\cdot|\right)$ is positive definite and $F\left[\Omega_0\right] = 0$, then indicator function χ_{Ω_0} minimizes functional

$$(1.12) \qquad F\left[\varphi\right] = \int\int f\left(|x-y|\right) \varphi(x) \varphi(y) \, \mathrm{d}x \, \mathrm{d}y$$

defined on the set of compactly supported functions. Then necessarily

$$\text{(1.13)} \qquad \int_{\Omega_0} f\left(|x-y|\right) \, \mathrm{d}y = 0$$

for all $x \in \mathbb{R}^n$. As a consequence, the second term in the last part of (1.11) is zero.

So if $F\left[\Omega_0\right] = 0$, then

$$\text{(1.14)} \qquad \ddot{F}\left[\Omega_0; V, V\right] = 2 \int_{\partial\Omega_0} \int_{\partial\Omega_0} f\left(|x-y|\right) v\left(x\right) v(y) \, \mathrm{d}s_x \, \mathrm{d}s_y,$$

for all $V \in C^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$. Here

$$\text{(1.15)} \qquad v\left(x\right) = \langle V, \eta \rangle \left(x\right)$$

denotes the normal component of vector field V on $\partial\Omega_0$.

2. Proof of Lemma 0.7

The proof is based on standard properties of Fourier transform and of Bessel functions. For them we refer to [18] and [19].

Let B be a ball of radius R centered at $x = 0$ and $F\left[B\right] = 0$ for the domain functional (0.1) with positive definite function $f\left(|\cdot|\right)$. Then

(2.1)

$$0 = \int_B \int_B f\left(|x-y|\right) \, \mathrm{d}x \, \mathrm{d}y = \int_0^\infty \int_{|\xi|=r} \left|\hat{\chi}_B\right|^2\left(\xi\right) \, \mathrm{d}s_\xi \, \mathrm{d}\mu\left(r\right) =$$

$$\left(2\pi R\right)^n \int_0^\infty \int_{|\xi|=r} \frac{J_{\frac{n}{2}}^2\left(R\left|\xi\right|\right)}{\left|\xi\right|^n} \, \mathrm{d}s_\xi \, \mathrm{d}\mu\left(r\right) = \left(2\pi R\right)^n \omega_{n-1} \int_0^\infty r^{-1} J_{\frac{n}{2}}^2\left(Rr\right) \, \mathrm{d}\mu\left(r\right),$$

here ω_{n-1} denotes the surface measure of the unit sphere.

Because function $\frac{J_{\frac{n}{2}}\left(t\right)}{t^{\frac{n}{2}}}$ is entire and μ is positive Borel measure of polynomial growth [16], equality in (2.1) is possible only if measure μ is discrete and

$$\text{(2.2)} \qquad \operatorname{supp} \mu \subset \left\{t | J_{\frac{n}{2}}\left(Rt\right) = 0\right\}.$$

Now, let us consider equality $\ddot{F}\left[B; V, V\right] = 0$. We have

$$\ddot{F}\left[B; V, V\right] = \int_{\partial B} \int_{\partial B} f\left(|x-y|\right) v\left(x\right) v\left(y\right) \, \mathrm{d}s_x \, \mathrm{d}s_y =$$

(2.3)

$$\int_0^\infty \int_{|\xi|=r} \left|\left(\chi_{\partial B} v\right)^{\wedge}\right|^2\left(\xi\right) \, \mathrm{d}s_\xi \, \mathrm{d}\mu\left(r\right).$$

Function $v \in L_2\left(\partial B\right)$ enables decomposition

$$(2.4) \qquad\qquad v\left(R\xi\right) = \sum_{k=0}^{\infty} c_k Y_k\left(\xi\right), \quad |\xi| = 1,$$

for spherical harmonic Y_k of order k [18].

    Then

$$\ddot{F}\left[B; V, V\right] = \int_0^{\infty} r^{n-1} \int_{|\xi|=1} \sum_{k=0}^{\infty} |c_k|^2 \left|\left(\chi_{\partial B} Y_k\right)^{\wedge}\right|^2 \left(\xi r\right) \, ds_\xi \, d\mu\left(r\right) =$$

$$(2.5)$$

$$= \int_0^{\infty} r^{n-1} \int_{|\xi|=1} \sum_{k=0}^{\infty} |a_k|^2 \frac{J_{\frac{n+2k-2}{2}}^2\left(Rr\right)}{r^{n-2}} Y_k^2\left(Rr\right) \, ds_\xi \, d\mu\left(r\right),$$

since

$$(2.6) \qquad \int_{|\xi|=1} Y_k\left(\xi\right) e^{irR\langle\xi,x\rangle} \, ds_\xi = const \cdot \frac{J_{\frac{n+2k-2}{2}}\left(Rr\right)}{r^{\frac{n-2}{2}}} Y_k\left(x\right)$$

and $const \neq 0$ [18].

    Bessel functions of different order has nonintersecting sets of zeroes [19], so due to relation (2.2) equality

$$(2.7) \qquad\qquad \int_{\partial B}\int_{\partial B} f\left(|x - y|\right) v\left(x\right) v\left(y\right) \, ds_x \, ds_y = 0$$

is valid if and only if

$$\int_{\partial B}\int_{\partial B} f\left(|x - y|\right) v\left(x\right) v\left(y\right) \, ds_x \, ds_y = |a_1|^2 \omega_{n-1} \int_0^{\infty} r J_{\frac{n}{2}}^2\left(Rr\right) \, d\mu\left(r\right) =$$

$$(2.8)$$

$$\int_{\partial B}\int_{\partial B} f\left(|x - y|\right) Y_1\left(x\right) Y_1\left(y\right) \, ds_x \, ds_y,$$

    where Y_1 is the first order spherical harmonic.

Remark 2.1. Function Y_1 has the form

$$(2.9) \qquad\qquad Y_1\left(x\right) = b_1 x_1 + b_2 x_2 + \ldots + b_n x_n = \langle b, x\rangle,$$

    i.e. on the surface of the ball B it coincides with a normal component of a constant vector field.

3. STABILITY OF THE BALL

In this Section we shall prove Theorem 0.5.

Because of Lemma 0.7, Remarks 0.4 and 2.1 it is sufficient to verify stability of the ball $\Omega = \Omega_0$ centered at $x = 0$ in direction of vector field V when function $v(x) = \langle V(x), x \rangle$ on the boundary $\partial \Omega$ is the first order spherical harmonic.

Lemma 3.1. *Suppose that point $x = 0$ is mass center of the domain Ω_t^V for all t close enough to zero and function $v(x) = \langle V(x), x \rangle$ on the boundary of the ball Ω_0 is the first order spherical harmonic . Then $v = 0$ on $\partial \Omega_0$.*

Proof. For all t close enough to zero and for any $i = \overline{1, n}$ equalities

$$\int_{\Omega_t^V} x_i \, \mathrm{d}x = 0$$

and

$$0 = \frac{d}{dt}|_{t=0} \int_{\Omega_t^V} x_i \, \mathrm{d}x = \int_{\partial \Omega_0} x_i v(x) \, \mathrm{d}s$$

holds. Consequently, $v = 0$ on the boundary $\partial \Omega_0$. $\qquad \square$

In general case, mass center $\bar{x}_t$ of the domain Ω_t^V is not zero. We consider the flow $\tilde{T}_t^V(x) := T_t^V(x) - \bar{x}_t$. This flow generates the family of domains $\tilde{\Omega}_t^V$ which coincide as rigid bodies with Ω_t^V and are centered at $x = 0$. Of course, the new vector field

$$\tilde{V}(t, x) := \frac{d}{dt} \tilde{T}_t^V(x)$$

is non–autonomous. Fortunately, shape derivative method is applied also in the case of non–autonomous vector fields $V \in \mathrm{C}^0\left(I, \mathrm{C}^1\left(\mathbb{R}^n, \mathbb{R}^n\right)\right)$, $I \subset \mathbb{R}$, $0 \in I$ [23]. Moreover, withdrawal of formula (1.14) in Section 1 remains the same in the case of $v(x) = \left\langle \tilde{V}(0, x), \eta(x) \right\rangle$.

4. SHAPE OPTIMIZATION AND THE POMPEIU PROBLEM

In this Section we shall prove Proposition 0.6.

Proof of a) Let equation (0.9) has bounded domain Ω as its own solution for some $f \neq 0$. By Bochner-Schwartz theorem [16] and because of spherical symmetry of $f(|\cdot|)$, there exists μ — positive Borel measure on $(0, \infty)$ of polynomial growth:

(4.1)
$$F[\Omega] = \int_{\Omega} \int_{\Omega} f(|x - y|) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^n} |\hat{\chi}_\Omega|^2(\xi) \, \mathrm{d}\mu(|\xi|) = \int_0^\infty \int_{|\xi| = r} |\hat{\chi}_\Omega|^2(\xi) \, \mathrm{d}s_\xi \, \mathrm{d}\mu(r),$$

where

$$(4.2) \qquad \hat{\chi}_\Omega(\xi) = \int_\Omega e^{i\langle \xi, x \rangle} \, dx.$$

Then equality

$$(4.3) \qquad F[\Omega] = 0$$

implies the existence of $\lambda \geq 0$:

$$(4.4) \qquad \int_{|\xi|=\lambda} |\hat{\chi}_\Omega|^2(\xi) \, ds_\xi = 0.$$

Consequently,

$$(4.5) \qquad \hat{\chi}_\Omega(\xi) = 0$$

for all $|\xi| = \lambda$. Because of $\hat{\chi}_\Omega(0) = \text{mes } \Omega > 0$, one has $\lambda > 0$.

Proof of b) Suppose that Ω has no the Pompeiu property. Then ([2], [5]) there exists $\lambda > 0$ equality (4.5) is true and, therefore,

$$(4.6)$$

$$0 = \int_{|\xi|=\lambda} |\hat{\chi}_\Omega|^2(\xi) \, ds_\xi = \int_{|\xi|=\lambda} \left[\int_\Omega \int_\Omega e^{i\langle \xi, x-y \rangle} \, dx \, dy \right] ds_\xi =$$

$$\int_\Omega \int_\Omega \left[\int_{|\xi|=\lambda} e^{i\langle \xi, x-y \rangle} \, ds_\xi \right] dx \, dy = (2\pi\lambda)^{\frac{n}{2}} \int_\Omega \int_\Omega \frac{J_{\frac{n-2}{2}}(\lambda |x-y|)}{|x-y|^{\frac{n-2}{2}}} \, dx \, dy,$$

here J_p denotes the Bessel function.

This will end the proof of Proposition 0.6.

5. Concluding remarks

1. Let F be a domain functional defined on the set of bounded domains $\Omega \subset \mathbb{R}^n$ and invariant under the rigid motions of $\mathbb{R}^n$. Suppose that for the ball B centered at $x = 0$ the condition

(*) if $\ddot{F}[B; V, V] = 0$, then function $v(x) = \langle V(x), x \rangle$ is first order spherical harmonic

holds.

Proposition 5.1. *If $\ddot{F}[B; V, V] \geq 0$ for all $V \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and the condition (*) holds, then the ball B is weakly stable minimum point of domain functional F in sense of Definition 0.3.*

2. Concerning the Schiffer problem. The result of Theorem 0.5 does not cover ones stated in [1, 13]. This is because spectral parameter λ in (0.11) may vary with the domain Ω. Though this more general situation
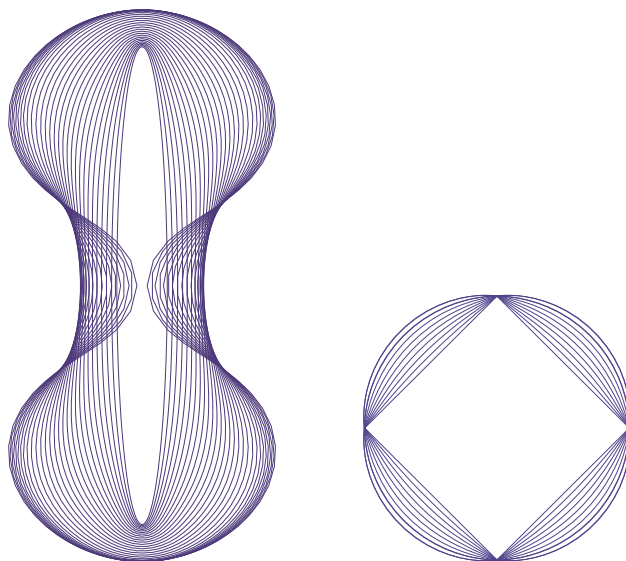
FIGURE 2.

may be considered on the basis of Lemma 0.7 the proof requires additional techniques.

3. Methods based on the shape derivative allow numerical experiments in the Pompeiu problem. In Fig. 2 evolution of the long thin ellipse and of the quadrate to the "figure" without the Pompeiu property in the shape antigradient direction is presented.

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